7.2 (continued)

can use Laplace transform to solve systems

$$x' = x + 2y$$
$$y' = 2x + y$$
$$x(0) = 1, \quad y(0) = 0$$

basic idea: transform, solve in s-domain, then inverse transform

$$\mathcal{L}\{x\} = X \qquad \mathcal{L}\{y\} = Y$$

$$\mathcal{L}\{x'\} = \mathcal{L}\{x + 2y\}$$

$$sX - x(0) = X + 2Y$$

$$sY - y(0) = 2X + Y$$

rewrite

$$(s-1)X - 2Y = 1$$
$$-2X + (s-1)Y = 0$$

solve for X, Y

$$\begin{bmatrix} s-1 & -2 \\ -2 & s-1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solve by row reduction, multiply by inverse of $\begin{bmatrix} s-1 & -2 \\ -2 & s-1 \end{bmatrix}$

or Cramer's rule.

$$X = \frac{\begin{vmatrix} 1 & -2 \\ 0 & s-1 \end{vmatrix}}{\begin{vmatrix} s-1 & -2 \\ -2 & s-1 \end{vmatrix}}$$

determinant of coefficient matrix w/ 1st column replaced by right side (because X is the 1st variable)

determinant of coefficient matrix

$$= \frac{s-1}{(s-1)^2 - 4} = \frac{s-1}{s^2-2s-3} = \frac{s-1}{(s-3)(s+1)}$$

$$= \frac{A}{s-3} + \frac{B}{s+1} = \cdots = \frac{1/2}{s-3} + \frac{1/2}{s+1}$$

$$x(t) = \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t}$$

$$* \; Y = \frac{\begin{vmatrix} s-1 & 1 \\ -2 & 0 \end{vmatrix}}{\begin{vmatrix} s-1 & -2 \\ -2 & s-1 \end{vmatrix}} = \frac{2}{(s-3)(s+1)} \quad \cdots \quad y(t) = \frac{1}{2}e^{3t} - \frac{1}{2}e^{-t}$$

7.3 Translation of Laplace Transform

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = F(s)$$

shift horizontally by a in s-domain: $F(s-a)$

$$F(s-a) = \int_0^\infty f(t)\,e^{-(s-a)t}\,dt$$

$$= \int_0^\infty \left[f(t)e^{at}\right]e^{-st}dt = \mathcal{L}\{f(t)e^{at}\}$$

multiplication of e^{at} in t-domain is a shift by a in s-domain.

from table: $\mathcal{L}\{1\} = \frac{1}{s}$

shift in s by a $\rightarrow$ multiplication by e^{at} in t

$F = \frac{1}{s-a} \rightarrow f = 1.e^{at} = e^{at}$ which agrees with table entries

same with $\sin(at) \rightarrow \frac{a}{s^2+a}$

$$\frac{a}{(s-c)^2+a} \rightarrow e^{ct}\sin(at)$$

<u>example</u> $y'' - 2y' + 5y = 8e^t$

$y(0) = 2, \quad y'(0) = 4$

$$s^2Y - sy(0) - y'(0) - 2(sY - y(0)) + 5Y = \frac{8}{s-1}$$

$$(s^2 - 2s + 5)Y = 2s + 4 - 4 + \frac{8}{s-1}$$

$$= 2s + \frac{8}{s-1} = \frac{2s^2 - 2s + 8}{s-1}$$

$$Y = \frac{2s^2 - 2s + 8}{(s-1)\underbrace{(s^2-2s+5)}}$$

if cannot be factored, put into $(s-a)^2 \pm b^2$

$$s^2 - 2s + 5 = s^2 - 2s + 1 + 4$$

$$= (s-1)^2 + 4$$

$$Y = \frac{2s^2 - 2s + 8}{(s-1)\left[(s-1)^2+4\right]} = \frac{A}{s-1} + \frac{Bs + C}{(s-1)^2 + 4}$$

$$2s^2 - 2s + 8 = A\left[(s-1)^2 + 4\right] + (Bs + C)(s-1)$$

$$\vdots$$

$$= (A+B)s^2 + (-2A + C - B)s + (5A - C)$$

$$A + B = 2 \qquad\qquad A = 2$$

$$-2A + C - B = -2 \quad \cdots \quad B = 0$$

$$5A - C = 8 \qquad\qquad C = 2$$

$$Y = \frac{2}{s-1} + \frac{2}{(s-1)^2 + 4}$$

$\frac{2}{s^2+4}$ shifted by 1

$\sin(2t)$ times e^t

$$y = 2e^t + e^t \sin(2t)$$

$\mathcal{L}\{f(t)e^{at}\} = F(s-a)$ 1st Translation Theorem

2nd Translation Theorem: $\mathcal{L}\{f(t-a)\} = ?$

assuming $f(t-a)=0$ for $t<a$